FLOW STABILITY OF A FLAT PLASTIC RING
WITH FREE BOUNDARIES

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The problem of two-dimensional unstable flow of an ideally plastic ring acted upon by internal pressure is formulated. The determination of the law of motion for the boundaries and of the time change of pressure is reduced to an ordinary nonlinear differential equation of the second order. For this equation a particular solution of the Cauchy problem is determined; this corresponds to a widening of the ring boundaries with a negative acceleration. For the field of initial velocities an estimate from above is available, expressed in terms of the original parameters. The very particular unstable flow obtained for an ideally plastic ring is also investigated with respect to stability to small harmonic perturbations of the velocity vector, the pressure, or the boundaries of the ring. It is shown that the fundamental flow is stable irrespective of the wave number. This result has been obtained by assuming that the inertial forces in the perturbed flow are small compared to the lasting ones.

1. A two-dimensional deformation problem is considered of a plastic ring acted upon by internal pressure P(t). There is no pressure at the outer boundary of the ring.

The components of the stress tensor σ_r, σ_θ, $\tau_{r\theta}$ and of the velocity vector v_r, v_θ satisfy in polar coordinates r, θ the equations of motion for a continuous medium outside the field of mass forces, namely,

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right);$$
$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = \rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right), \tag{1.1}$$

where ρ denotes the density of the material.

The equation of state of the plastic medium is adopted in the form

$$(\sigma_\theta - \sigma_r)^2 + 4\tau_{r\theta}^2 = 4K^2, \tag{1.2}$$

where $K > 0$ is the plasticity constant. Moreover, it is also assumed that the principal directions of the stress tensor are identical at every point of the medium with the principal directions of the deformation rate tensor whose components are expressed by means of the following equalities:

$$\varepsilon_r = \frac{\partial v_r}{\partial r}; \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r};$$
$$2\gamma_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}. \tag{1.3}$$

The assumption that the medium is incompressible yields the well-known relation

$$\varepsilon_r + \varepsilon_\theta = 0. \tag{1.4}$$

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The dynamic condition

$$\sigma_r = -P(t) \text{ for } r=R_1, \quad \sigma_r = 0 \text{ for } r=R_2 \tag{1.5}$$

on the boundaries of the ring must be satisfied, as well as the kinematic condition,

$$\frac{dR_j}{\partial t} = v_r \quad \text{for} \quad r=R_j \ (j=1,2). \tag{1.6}$$

The ring is formed by two concentric circles of radii R_1 and R_2 ($R_1 < R_2$).

In the case of axial symmetry of the ring flow the solution of the boundary-value problem (1.1)-(1.6) is given by the following expressions:

$$\tau_{r\theta}=0; \ v_\theta=0; \ \gamma_{r\theta}=0; \tag{1.7}$$

$$\sigma_r = -(\rho\dot{a}+2K)\ln R_2/r + \frac{1}{2}\rho a^2\left(r^{-2}-R_2^{-2}\right); \tag{1.8}$$

$$\sigma_\theta=\sigma_r+2K; \ v_r=ar^{-1}; \ \varepsilon_\theta=-\varepsilon_r=ar^{-2}.$$

In the above one has

$$P = (\rho\dot{a}+2K)\ln R_2/R_1 - \frac{1}{2}\rho a^2\left(R_1^{-2}-R_2^{-2}\right); \tag{1.9}$$

$$a(t)=\dot{R}_1R_1; \ \dot{R}_1R_1=\dot{R}_2R_2;$$

$$R_j=R_{jo}, \ \dot{R}_j=V_{jo} \text{ for } t=0; \tag{1.10}$$

R_{jo}, V_{jo} are the initial radius and the widening rate of the ring boundaries, respectively; the dot signifies the differentiation with respect to time $t \geq 0$. By virtue of the kinematic condition (1.6) one has for the initial values the ratios

$$R_{20}/R_{10}=V_{10}/V_{20}, \tag{1.11}$$

and in view of incompressibility at any time instant one has the equality

$$R_2^2-R_1^2=R_{20}^2-R_{10}^2. \tag{1.12}$$

If the pressure function P(t) is known then the relations (1.9)-(1.12) determine the flow of the widening ring. The problem of finding one of the radii $R_j(t)$ reduces to the Cauchy problem for a nonlinear differential equation of the second order whose exact solution involves basic mathematical difficulties.

However, the inverse problem can be considered, that is, one can construct a rule for the motion of the ring boundaries which would satisfy the conditions (1.10)-(1.12), the pressure change within the ring being obtained from (1.9) if $R_1(t)$ and $R_2(t)$ are given.

A simple verification shows that if the radii satisfy the expressions

$$R_j = R_{j0}\left(1+2\frac{V_{j0}}{R_{j0}}t\right)^{1/2}; \quad \dot{R}_j = V_{j0}\frac{R_{j0}}{R_j}, \tag{1.13}$$

then the conditions (1.10)-(1.12) are fulfilled.

Then bearing in mind that $\dot{a}(t)=0$ the pressure within the ring is obtained from (1.9) by the formula

$$P = 2K\ln R_2/R_1 - \alpha_0\left(R_1^{-2}-R_2^{-2}\right); \quad \alpha_0 = \frac{1}{2}\rho V_{10}^2R_{10}^2. \tag{1.14}$$

Since at the initial time instant one has $P(0)=P_0 \geq 0$ therefore in accordance with (1.14) one has to impose a constraint on the initial widening rate of the inner boundary of the ring, namely,

$$V_{10} \leqslant V_0; \quad V_0 = 2\left[\frac{K\ln\varkappa_0}{\rho\left(1-\varkappa_0^{-2}\right)}\right]^{1/2};$$

$$\varkappa_0 = R_{20}/R_{10}. \tag{1.15}$$

If $V_{10}=V_0$, then $P_0=0$ and the ring flow takes place inertially in accordance with the given velocity field. If, however, $V_{10}< V_0$, the exact solution (1.13) and (1.14) admits the following interpretation: at the initial instant, in the plastic ring with radii $R_{j0}(R_{10}< R_{20})$ there is given a velocity field $V_{j0}(V_{10}> V_{20})$ within which there is a specified pressure $P_0> 0$ indispensable for plastic deformation of the ring. For $t\geq 0$ the ring widens and the flow rate of the ring boundaries diminishes in the course of time. Moreover, the pressure within the ring is reduced to zero for $t\to\infty$.

2. The constructed particular solution of an unstable flow of the plastic ring is now investigated as regards stability relative to small perturbations of the flow rate, of pressure, and of the ring boundaries. Two cases can be distinguished: $V_{10}=0$, $V_{10}< V_0$.

Let

$$v_r' = v_r^0 + v_r^*;\quad v_\theta' = v_\theta^0 + v_\theta^*;\qquad \sigma' = \sigma^0 + \sigma^*;\quad R_j' = R_j + R_j^*, \tag{2.1}$$

where the fundamental flow carries the superscript zero; v'_r, v'_θ, are the particle velocities in the perturbed flow; R'_j is the radius of the perturbed boundaries; σ' is the arithmetic mean of fundamental stresses of the perturbed flow.

Since the elementary perturbations v^*_r, v^*_θ, σ^*, R^*_j are slow one can expect that the motion of the ring with perturbed boundaries differs only slightly from the flow of the concentric ring. Therefore, it can be assumed [1] that the principal direction of the flow in the perturbed motion which corresponds to the tangent direction to the perturbed ring surface makes only a small angle θ' with the principal direction of the unperturbed ring. In accordance with the transformation formulas for the components of the stress tensor and of the strain rate tensor in polar coordinates, as well as by using the method of [2] for linearization of stresses and strains, one obtains expressions for the components of the stress tensor of the perturbed motion in the form

$$\sigma_r' = \sigma_r^0 + \sigma^*;\quad \sigma_\theta = \sigma_\theta^0 + \sigma^*;$$
$$\tau_{r\theta}' = \tau^0\left[\frac{\partial v_\theta^*}{\partial r} - \frac{1}{r}\left(v_\theta^* - \frac{\partial v_r^*}{\partial \theta}\right)\right];$$
$$\tau^0 = K/(\varepsilon_\theta^0 - \varepsilon_r^0). \tag{2.2}$$

Since the medium under consideration is incompressible one obtains from (1.3), (1.4), and (2.1)

$$\frac{\partial}{\partial\theta}(v_\theta^*) + \frac{\partial}{\partial r}(r v_r^*) = 0.$$

Hence it follows that there exists a sufficiently smooth function $\chi(r, \theta, t)$, such that the relations are valid,

$$v_\theta^* = \frac{\partial\chi}{\partial r};\quad v_r^* = -\frac{1}{r}\frac{\partial\chi}{\partial\theta}. \tag{2.3}$$

By linearizing Eq. (1.1) and using (2.1)-(2.3) and ignoring the inertia terms in the perturbed equations of motion one obtains

$$\frac{\partial\sigma^*}{\partial r} + r^{-1}\frac{\partial\Phi}{\partial\theta} = 0;$$
$$r^{-1}\frac{\partial\sigma^*}{\partial\theta} + 2r^{-1}\Phi + \frac{\partial\Phi}{\partial r} = 0. \tag{2.4}$$

In the above one has

$$\Phi = \tau^0\left[\frac{\partial^2\chi}{\partial r^2} - \frac{1}{r}\left(\frac{\partial\chi}{\partial r} + \frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2}\right)\right]. \tag{2.5}$$

It should be mentioned that the assumption with regard to the smallness of the inertia terms in the perturbed equation of motion is more justified for the particular solution under consideration in which one had the estimate from above for the initial velocities field than, for example, in [1, 2].

Having transferred the stresses σ'_r, σ'_θ, $\tau'_{r\theta}$ onto the unperturbed boundaries the boundary conditions (1.5) on the curvilinear surfaces of the perturbed ring are given by

$$\sigma_\theta \cos(x', \nu) + \tau'_{r\theta}\cos(y', \nu) = 0 \quad \text{for} \quad r = R_j; \tag{2.6}$$
$$\tau'_{r\theta}\cos(x', \nu) + \sigma'_r \cos(y', \nu) = -P' \quad \text{for} \quad r = R_1;$$
$$\tau'_{r\theta}\cos(x', \nu) + \sigma'_r \cos(y', \nu) = 0 \quad \text{for} \quad r = R_2,$$

where (x', y') is a rectangular coordinate system with origin on the unperturbed surface of the ring; ν is the outer normal to the perturbed boundary.

Since $\cos(x', \nu) = \sin(y', \nu)$, where $\tan(y', \nu) = \frac{1}{R_j}\frac{\partial R_j^*}{\partial \theta}$, and since the perturbation is small one obtains

$$\cos(x', \nu) \simeq \frac{1}{R_j}\frac{\partial R_j^*}{\partial\theta}; \qquad \cos(y', \nu) \simeq 1. \tag{2.7}$$

Consequently, it follows from (2.1)-(2.3), (2.6)-(2.7), and using the parameter relation of the fundamental flow that for $V_{10} < V_0$ one has

$$\sigma_\theta^0 \frac{1}{R_j}\frac{\partial R_j^*}{\partial\theta} + \Phi = 0 \quad (r = R_j); \tag{2.8}$$
$$\sigma^* = 2K\left(\frac{R_1^*}{R_1} - \frac{R_2^*}{R_2}\right) + 2\alpha_0\left(R_2^{-2}\frac{R_2^*}{R_2} - R_1^{-2}\frac{R_1^*}{R_1}\right) \quad (r = R_1);$$
$$\sigma^* = 0 \ (r = R_2).$$

If $V_{10} = V_0$, then $\sigma^* = 0$ on both boundaries of the ring.

The kinematic condition (1.6) for the perturbed flow is given by (see [3])

$$\frac{\partial R_j^*}{\partial t} + r^{-1}\frac{\partial\chi}{\partial\theta} - \frac{\partial v_r^0}{\partial r}R_j^* = 0 \quad (r = R_j). \tag{2.9}$$

3. The variables are now changed by

$$x = \frac{V_{10}}{R_{10}}t; \qquad y = \ln r/R_1 \tag{3.1}$$

and perturbation of the ring boundaries and of the corresponding pressure is considered,

$$R_j^* = \xi_j(x)\sin\omega\theta; \qquad \sigma^* = \varphi(x, y)\sin\omega\theta. \tag{3.2}$$

Then conforming to the shape of the boundary conditions the solution for the function $\chi(r, \theta, t)$ is sought in the form

$$\chi = \psi(x, y)\cos\omega\theta, \tag{3.3}$$

where $\omega = 0, 1, 2, \ldots$ is the wave number of the harmonic perturbation under consideration.

The dimensionless quantities

$$\bar\psi = \psi/V_{10}R_{10}; \quad \bar\varphi = \varphi/\rho V_{10}^2; \quad \bar K = K/\rho V_{10}^2; \quad \bar P = P/\rho V_{10}^2; \tag{3.4}$$
$$\bar R_j = R_j/R_{10}; \quad \bar\xi_j = \xi_j/R_{10}; \quad \bar V_{10} = V_{10}/V_0; \quad \varkappa = R_2/R_1.$$

are not introduced. By substituting (3.1)-(3.4) into (2.4) and (2.5) one obtains a system of partial differential equations with constant coefficients for the functions $\psi(x, y)$ and $\varphi(x, y)$, respectively (the bar is now omitted for the dimensionless quantities):

$$\frac{\partial^4\psi}{\partial y^4} + 2(\omega^2 - 2)\frac{\partial^2\psi}{\partial y^2} + \omega^4\psi = 0; \tag{3.5}$$

$$\omega\varphi = -\frac{1}{2}K\left(2 + \frac{\partial}{\partial y}\right)\left[\frac{\partial^2\psi}{\partial y^2} - 2\frac{\partial\psi}{\partial y} + \omega^2\psi\right]. \tag{3.6}$$

The boundary conditions (2.8) now become

$$\sigma_\theta^0 \frac{\omega}{R_j}\xi_j + \frac{1}{2}K\left(\frac{\partial^2\psi}{\partial y^2} - 2\frac{\partial\psi}{\partial y} + \omega^2\psi\right) = 0 \quad (y = \ln R_j/R_1); \tag{3.7}$$

$$\varphi = 2K\left(\frac{\xi_1}{R_1} - \frac{\xi_2}{R_2}\right) + R_2^{-2}\frac{\xi_2}{R_2} - R_1^{-2}\frac{\xi_1}{R_1} \quad (y = 0); \tag{3.8}$$
$$\varphi = 0 \ (y = \ln\varkappa).$$

Then in the case of $V_{10} < 1$ one has

$$\sigma_\theta^0 = 2K - P; \quad P = 2K \ln \varkappa - R_2^{-2}(\varkappa^2 - 1) \quad (y = 0); \qquad (3.9)$$
$$\sigma_\theta^0 = 2K \quad (y = \ln \varkappa).$$

If $V_{10} = 1$, then $\sigma_\theta^0 = 2K$, $\varphi = 0$ on both boundaries of the ring. It follows from the kinematic condition (2.9) that

$$\frac{d\xi_j}{dx} + R_j^{-2}\xi_j = \omega R_j^{-1}\psi \qquad (y = \ln R_j/R_1) \qquad (3.10)$$

with the initial condition

$$\xi_j = \xi_j^0 \quad \text{for} \quad x = 0. \qquad (3.11)$$

4. Thus, the relations (3.5)-(3.11) specify a complete problem for the behavior of the harmonic perturbation of the velocity vector, pressure, and the ring boundaries with the inertia forces ignored in the pertubed flow. The equations (3.5) and (3.6) are solved by using the form

$$\psi = \sum_{i=1}^{4} C_i \exp(n_i y); \qquad (4.1)$$
$$\omega\varphi = -\frac{1}{2} K \sum_{i=1}^{4} A_i C_i \exp(n_i y),$$

where

$$n_i = \pm\ [2 - \omega^2 \pm 2(1 - \omega^2)^{1/2}]^{1/2}, \qquad (4.2)$$

$n_1 = -n_2$, $n_3 = -n_4$ are the roots of the characteristic equation

$$n^4 + 2(\omega^2 - 2)\, n^2 + \omega^4 = 0,$$
$$A_i = N_i(n_i + 2), \quad N_i = n_i^2 - 2n_i + \omega^2, \quad (i = 1,\ 4).$$

In the case of $\omega = 1$ the characteristic equation possesses multiple roots $n_1 = n_3 = 1$, $n_2 = n_4 = -1$. Without dwelling specifically on this case one can only mention that for $\omega = 1$ and $\omega > 1$ the finite results coincide. Subsequently, n_i will be considered as being different.

By substituting the relations (4.1) into the boundary conditions (3.7) and (3.8) an algebraic system of linear equations of the fourth order is obtained for the time functions $C_i(x)$:

$$\sum_{i=1}^{4} N_i C_i = f_1, \quad f_1 = -4\omega R_2^{-1}\varkappa\left(1 - \frac{P}{2K}\right)\xi_1; \qquad (4.3)$$
$$\sum_{i=1}^{4} N_i C_i \varkappa^{n_i} = f_2, \quad f_2 = -4\omega R_2^{-1}\xi_2;$$
$$\sum_{i=1}^{4} A_i C_i = f_3, \quad f_3 = -4\omega R_2^{-1}\left[\varkappa\xi_1 - \xi_2 + \frac{R_2^{-2}}{2K}(\xi_2 - \varkappa^2\xi_1)\right]; \qquad \sum_{i=1}^{4} A_i C_i \varkappa^{n_i} = 0.$$

In the above $f_m(x)$, $m = 1, 3$ are the functions appearing in (3.7)-(3.9) which are given in terms of R_2 and $\varkappa$ in the case of $V_{10} < 1$.

Using the relations

$$\frac{d\xi_j}{dx} = \frac{d\eta_j}{dR_2} R_2^{-1}; \quad R_2 = (\varkappa_0^2 + 2x)^{1/2}$$
$$\eta_j(R_2) = \xi_j(x),$$

which hold by virtue of (1.13) one obtains from (3.10) and (3.11) a system of ordinary differential equations of the first order,

$$\frac{d\eta_1}{dR_2} + R_2^{-1}\varkappa^2\eta_1 = \omega\varkappa \sum_{i=1}^{4} C_i; \qquad (4.4)$$
$$\frac{d\eta_2}{dR_2} + R_2^{-1}\eta_2 = \omega \sum_{i=1}^{4} C_i \varkappa^{n_i}$$

with the initial conditions

$$\eta_i(\varkappa_0) = \eta_j^0 \quad \text{for } R_2 = \varkappa_0.$$

Since $R_2 = R_1 + \delta$, where δ denotes the current ring thickness therefore $\varkappa = 1 + \varepsilon$, $\varepsilon = \delta/R_1$ and for a widening ring one has $\varepsilon \to 0$. The method for finding a solution for ordinary differential equations containing a small parameter ε is sufficiently well known (see, for example, [4]). It is noted that the parameter ε appears in (4.4) in a regular manner.

The first terms of the asymptotic series

$$\eta_j = \sum_{q=0}^{\infty} \eta_{jq}(R_2)\, \varepsilon^q; \qquad f_m = \sum_{q=0}^{\infty} f_{mq}(R_2)\, \varepsilon^q$$

are determined by using the condition $\varepsilon = 0$ ($\varkappa = 1$). It follows from (4.3) and (4.4) that $\eta_{10} = \eta_{20} = \eta_0$. The latter indicates that if during the main motion the ring transforms into a circle with zero wall thickness then the amplitude of perturbation of the boundaries is unique. Then from (4.3) one has that $f_{10} = f_{20}$, $f_{30} = 0$ and to obtain an asymptotic solution the cases $V_{10} < 1$, $V_{10} = 1$ are identical for $\varepsilon = 0$. Hence the system (4.3) is of rank two. By virtue of (4.2) and making use of the relations

$$N_1N_2 = N_3N_4 = N_1N_4 = N_2N_3 = 0; \quad N_1N_3 = 4\omega^2(2 - n_1 - n_3);$$
$$N_2N_4 = 4\omega^2(2 - n_1 - n_4);$$
$$A_1 = A_3; \quad A_2 = A_4,$$

one obtains $\sum_{i=1}^{4} C_i = 0$.

It follows from (4.4) that

$$\eta_0(R_2) = CR_2^{-1}, \qquad C = \text{const},$$

and therefore $\eta_0(R_2) \to 0$ for $R_2 \to \infty$. It can similarly be shown that $\eta_{j1}(R_2) \to 0$ for $R_2 \to \infty$ for any $\omega > 0$.

The question whether $\eta_{jq}(R_2) \to 0$ can be considered further for any $q \geq 2$ remains unanswered. It is more expedient to calculate higher approximations numerically.

As it was said above, the asymptotics of the amplitude of harmonic perturbation of the ring boundaries with accuracy up to the second order of smallness with regard to t diminishes to zero ($\eta_j(R_2) \to 0$), if $R_2 \to \infty$. It follows from (2.3) and (4.1) that perturbation of the velocity vector and of pressure approaches zero for $\varepsilon \to 0$. Consequently, the fundamental unsteady flow of a plastic ring is stable as regards small harmonic perturbations of the velocity vector, of pressure, or of boundaries for any wave number provided the initial parameters of the problem satisfy the conditions (1.15) and (4.10).

It is known [3] that in the case of a thin ring of ideal incompressible fluid which expands inertially the harmonic perturbations of the ring boundaries grow without bounds. One can also observe the instability of a flow of ideal fluid in a ring acted upon by variable internal pressure [5]. Such instability can be observed for any $\omega > 0$.

If we introduce into our considerations the forces of internal resistance of the medium, say, plastic, in the presence of an estimate from above for the initial velocity field of the fundamental flow then under the assumption of small inertia forces in the perturbed motion the main flow investigated above is stable relative to small harmonic perturbations of the velocity vector, of pressure, or of ring boundaries for any wave number.

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